Category, space, type - Benjamin Antieau
01. What is a topological space?

The definition of topological spaces was given by Hausdorff in 1914. It aims to capture some of the qualitative notions of geometry and analysis divorced from quantities like distance and angle. In the intervening years, it has come to be adapted by the mathematical community as the correct definition giving a substrate upon which to build more complicated structures. One of the themes of this course is to play with this definition and see how it relates to other possible definitions.

The terminology of topological spaces is motivated by the example of the 'space' $\mathbf{R}$ of real numbers, the real number line. Certain subsets, like $(0,1) \subseteq \mathbf{R}$, are considered to be open, while other, like $[0,1]$, are considered to be closed. This terminology is familiar from school. The closed subsets are characterized by being *closed* under taking limits of sequences. For example, the sequence $\frac{1}{n}$ lies entirely within $(0,1)$, but its limit point 0 is not in $(0,1)$, whereas it does certainly stay inside the larger subset $[0,1]$.

Alternatively, the open subsets are ones which are closed under small perturbations. A point inside an open subset like $(0,1)$, if it is wiggled a very small amount will stay inside $(0,1)$. The amount of wiggle allowed will depend on the point. But, if one wiggles 0 any amount, it will probably fall off of $[0,1]$.

Taking intuition from this example, we will study topological spaces which do not arise necessarily from the study of classical geometry or calculus. The motivation for the development of this theory historically is due to these sources. Especially troubling was the problem of determining which functions could be approximated by Fourier series. This required notions of convergence for sequences not in $\mathbf{R}$ but in 'spaces' whose points were functions on other spaces. These objects look rather exotic from the perspective of euclidean or cartesian geometry.

We have the following requirements for a good theory of 'space':

(1) it should distinguish between $\mathbf{R}^m$ and $\mathbf{R}^n$ for $m \neq n$;

(2) it should support a rich palette of examples, both classical and exotic;

(3) it should include the standard examples from analysis, including $\mathbf{R}^n$ but also 'spaces' of continuous or differentiable functions;

(4) it should give meaning to the closedness of $[0, 1] \subseteq \mathbf{R}$ and the openness of $(0, 1) \subseteq \mathbf{R}$;

(5) it should support a notion of convergence, as in $\lim_{n\to\infty} \frac{1}{n} = 0$;

(6) it should explain how $[0, 1]$ is finite, whereas $[0, \infty)$ is not;

(7) it should have a good notion of functions between 'spaces';

(8) it should be able to see the difference between $(0, 1)$ and $(0, 1) \cup (2, 3)$ in that the former should be 'connected' and the latter should be 'disconnected'

(9) it should have a sense of completeness and know how to pass from $\mathbf{Q}$ to $\mathbf{R}$.

Many of these requirements are satisfied by the related theory of metric spaces, which is discussed below. However, the following definition turns out to be more flexible and more suitable as a notion of 'space' for all of mathematics.

`Definition 1.1 (Topological space).` A topological space is a pair $(X, \mathcal{U})$ where X is a set and $\mathcal{U} \subseteq \mathbf{P}(X)$ is a set of subsets of X satisfying the following conditions:

(i) $\emptyset, X \in \mathcal{U}$;

(ii) for every finite set $U_1, \ldots, U_n$ of subsets in $\mathcal{U}$, the intersection $U_1 \cap \cdots \cap U_n$ is in $\mathcal{U}$;

(iii) for every subset $\{U_i\}_{i\in I} \subseteq \mathcal{U}$, the union $\cup_{i\in I} U_i$ is in $\mathcal{U}$.

The elements of $\mathcal{U}$ are called opens in X or open subsets of X.

`Definition 1.2 (Closed sets).` Let $(X, \mathcal{U})$ be a topological space. A subset $Y \subseteq X$ is called closed if $X \setminus Y$ is open. The closed subsets of X satisfy similar axioms to those appearing in Definition 1.1 except they are closed under arbitrary intersections but only finite unions. The open subsets determine the closed subsets and vice versa, so an alternative, equivalent way to define a topological space is via its closed subsets.

`Example 1.3 (The discrete topology).` The pair $(X, \mathcal{P}(X))$ is a topological space; this is the discrete topology on X. Every subset of X is declared to be open. We will write X^δ for a set X equipped with the discrete topology.

`Example 1.4 (The trivial topology).` The pair $(X, \{\emptyset, X\})$ is a topological space, the trivial topology on X. Only $\emptyset$ and X are open. Let X^{triv} denote a set X with the trivial topology.

`Definition 1.5 (Metric spaces).` A metric space is a pair (X, d) where X is a set and $d\colon X \times X \to \mathbf{R}$ is a function such that

(a) $d(x, y) = 0$ if and only if $x = y$,

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$, and

(c) $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The last condition is called the triangle inequality. Say that $U \subseteq X$ is d-open if it satisfies the following property: for each $u \in U$ there exists an $\epsilon > 0$ such that if $d(u, x) < \epsilon$, then $x \in U$. Let $\mathcal{U}$ be the collection of d-open subsets. Then, $\mathcal{U}$ defines a topology on the set X as is shown in Lemma 1.9 below.

`Example 1.6 (Euclidean space).` In this course, we will consider $\mathbf{R}^n$ as a metric space via the euclidean distance function and as a topological space via the topology associated to d. Recall that if $x, y \in \mathbf{R}^n$ have coordinates $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, then the euclidean distance between them is

$$d(x, y) = +\sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}.$$

The term $(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2$ is always a nonnegative real number. It is zero if and only if $x_i = y_i$ for $i = 1, \ldots, n$, i.e., if and only if $x = y$. This is condition (a) for a metric. We also have that $(y_i - x_i)^2 = (x_i - y_i)^2$ for all i, so condition (b) holds. Condition (c) is more complicated and follows from the Cauchy–Schwarz inequality

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leqslant \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$$

for real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$.

The open subsets of $\mathbf{R}^n$ are unions of small open disks. If (X, d) is a metric space, $x \in X$ is a point, and $\epsilon > 0$ is a positive real number, then let $B_\epsilon(x) \subseteq X$ be the set of all points $y \in X$ such that $d(x, y) < \epsilon$. This is the open ball of radius ϵ around x. It is a d-open subset by the triangle inequality.

`Exercise 1.7.` Show that if (X, d) is a metric space, then the open balls $B_\epsilon(x)$ are d-open subsets.

`Exercise 1.8.` (i) Find and understand a proof of the Cauchy–Schwarz inequality, for example by consulting any textbook on real analysis.

(ii) Use the Cauchy–Schwarz inequality to prove the triangle inequality for the euclidean distance function.

`Lemma 1.9.` *Let (X, d) be a metric space. If $\mathcal{U}$ is the set of d-open subsets of X, then $(X, \mathcal{U})$ is a topological space.*

Proof. The empty set and X itself are both d-open. If $U_1, \ldots, U_n$ are d-open subsets and $u \in U_1 \cap \cdots \cap U_n$, then there are real numbers $\epsilon_1, \ldots, \epsilon_n > 0$ such that if $d(u, x) < \epsilon_i$, then $x \in U_i$ for $i = 1, \ldots, n$. Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. Then, if $d(u, x) < \epsilon$, we must have $x \in U_1, \ldots, U_n$, or in other words $x \in U_1 \cap \cdots \cap U_n$. It follows that $U_1 \cap \cdots \cap U_n$ is d-open. If $\{U_i\}_{i \in I}$ is a collection of d-open subsets, let $U = \cup_{i \in I} U_i$. If $u \in U$, then $u \in U_i$ for some $i \in I$. Pick $\epsilon > 0$ such that if $d(u, x) < \epsilon$, then $x \in U_i$. Then, if $d(u, x) < \epsilon$, $x \in U$. It follows that U is d-open. We have verified conditions (i), (ii), and (iii) for the collection $\mathcal{U}$ of d-open subsets, so $(X, \mathcal{U})$ is a topological space. $\square$

`Remark 1.10 (Metrizable spaces).` A topological space $(X, \mathcal{U})$ is said to be metrizable if there exists a metric d on X such that $\mathcal{U}$ is equal to the collection of d-open sets. Note that if $\alpha > 0$ is any positive real number, then if $(X, \mathcal{U})$ is metrized by a metric d, then it is also metrized by αd. The open sets do not change under scaling. One reason that topological spaces are favored over metric spaces is that specifying a metric seems to be overkill. Of course, for other, geometric problems, the specification of a metric is crucial.

`Example 1.11 (The other topology on two points).` Consider the topology on $\{0, 1\}$ with $\mathcal{U} = \{\emptyset, \{0\}, \{0, 1\}\}$. This is neither the trivial nor the discrete topology. It is also not metrizable as Exercise 1.13 shows. In the literature, this is called the Sierpiński space.

`Exercise 1.12.` Write down all topologies on the set $\{1, 2, 3\}$.

`Exercise 1.13.` Show that if $(X, \mathcal{U})$ is a finite metrizable topological space, then $\mathcal{U}$ is the discrete topology on X.

`Example 1.14 (The cofinite topology).` Let X be a set and let $\mathcal{U} \subseteq \mathbf{P}(X)$ be the set of subsets $U \subseteq X$ such that $X \setminus U$ is a finite set (such subsets are called cofinite) together with the empty set. Then, $(X, \mathcal{U})$ is a topological space.

`Remark 1.15 (Infinite intersections).` The definition of a topological space requires that the open subsets are closed under arbitrary unions but only finite intersections. In $\mathbf{R}$ we see that that open subsets are unions of open intervals of the form (a, b). However, consider the family $\{(-\frac{1}{n}, \frac{1}{n})\}_{n \geqslant 1}$ of open subsets of $\mathbf{R}$. This family has intersection

$$\bigcap_{n \geqslant 1} (-\tfrac{1}{n}, \tfrac{1}{n}) = \{0\},$$

which is not open in $\mathbf{R}$. Some topological spaces have the property that their open subsets are closed under arbitrary intersections. These are called Alexandrov spaces.

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