Category, space, type - Benjamin Antieau
Problem set 03. Birkhoff's theorem

Problem 3.1. *Let L be a distributive lattice and let $x \in L$. Show that x is join-irreducible if and only if the following condition holds: if $x \leqslant y \vee z$ for $y, z \in L$, then $x \leqslant y$ or $x \leqslant z$.*

Problem 3.2. *Let $f\colon L \to M$ be a bounded lattice homomorphism between bounded distributive lattices. Show that f induces a poset morphism $\operatorname{Spec}(f)\colon \operatorname{Spec} M \to \operatorname{Spec} L$ as follows. If $y \in \operatorname{Spec} M$, then y is a join-irreducible element of M. Let $\operatorname{Spec}(f)(y) = \min\{x \in L\colon y \leqslant f(x)\}$.*

(i) *Show that the minimum is indeed in $\operatorname{Spec} L$, i.e., that it is join-irreducible.*

(ii) *Show that if $y \leqslant z$ in $\operatorname{Spec} M$, then $\operatorname{Spec}(f)(y) \leqslant \operatorname{Spec}(f)(z)$ in $\operatorname{Spec} L$.*

Problem 3.3. *Show that with the construction of Problem 3.2* Spec *(restricted to finite distributive lattices) defines a functor $\mathbf{Dist}^{\mathrm{fin}} \to \mathbf{Pos}^{\mathrm{fin,op}}$.*

Problem 3.4. *Let P be a poset. Let $\mathcal{D}_P \subseteq \mathbf{P}(P)$ be the subset of downsets. Show that $\mathcal{D}_P$ is a bounded distributive lattice.*

Problem 3.5. *Let $f\colon P \to Q$ be a morphism of posets. Show that $f^{-1}\colon \mathcal{D}_P \to \mathcal{D}_Q$ is a bounded lattice homomorphism. Write $\mathcal{D}_f$ for this morphism.*

Problem 3.6. *Show that $\mathcal{D}$ defines a functor $\mathbf{Pos}^{\mathrm{fin}} \to \mathbf{Dist}^{\mathrm{fin,op}}$.*

Problem 3.7. *Prove Birkhoff's theorem (Theorem 13.22) by showing that* Spec *and $\mathcal{D}$ are inverse equivalences of categories.*

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