Category, space, type - Benjamin Antieau
Problem set 02. Compactness

`Problem 1.1 (Tube lemma).` *Let X and Y be topological spaces and assume that X is compact. Fix $y \in Y$ and suppose that $U \subseteq X \times Y$ is an open subset containing $X \times \{y\}$. Show that there is an open subset $V \subseteq Y$ containing y such that U contains $X \times V$.*

`Problem 1.2 (Compactness of products).` *Let X and Y be compact topological spaces. Show that $X \times Y$ is compact. Conclude that if $X_1, \ldots, X_n$ are compact topological spaces, then $\prod_{i=1}^n X_i$ is compact.*

`Problem 1.3 (Finite intersection property).` *Let X be a set and let $\mathcal{C} \subseteq \mathbf{P}(X)$ be a set of subsets of X. Say that $\mathcal{C}$ has the finite intersection property if for every sequence $C_1, \ldots, C_n$ of elements of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.*

Show that X is compact if and only if for every collection $\mathcal{C}$ of closed subsets of X which satisfies the finite intersection property the intersection

$$\bigcap_{C \in \mathcal{C}} C$$

is nonempty.

`Problem 1.4 (Compactness of closed intervals).` *Come up with the "real analysis" proof of the fact that $[0, 1] \subseteq \mathbf{R}$ is compact.*

`Problem 1.5 (Compactness and subbasis).` *Let $\mathcal{B}$ be a subbasis for a topology $\mathcal{U}$ on X. Suppose that every open cover $\mathcal{C}$ of X such that $\mathcal{C} \subseteq \mathcal{B}$ has a finite subcover. Show that X is compact.*

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