Category, space, type - Benjamin Antieau
Problem set 01. Closure and interior

Problem 1.1. *Let X be a topological space. Show that every subset $A \subseteq X$ is contained in a smallest closed subset, written $\overline{A}$ and called the closure of A.*

Problem 1.2. *Let X be a topological space. Show that every subset $A \subseteq X$ contains a largest open subset A°, the interior of A.*

Problem 1.3. *Let X be a topological space and let $A \subseteq X$ be a subset. Prove the following identities:*

(i) $X \setminus (X \setminus A)^\circ = \overline{A}$;

(ii) $X \setminus \overline{X \setminus A} = A^\circ$.

Problem 1.4. *Let X be a topological space and let $A, B \subseteq X$ be subsets. Establish the Kuratowski closure axioms:*

(a) $\overline{\emptyset} = \emptyset$;

(b) $A \subseteq \overline{A}$;

(c) $\overline{\overline{A}} = \overline{A}$;

(d) $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$.

Definition 1.5. A Kuratowski closure operator on a set X is a function $C \colon \mathbf{P}(X) \to \mathbf{P}(X)$ which satisfies the Kuratowski closure axioms, i.e., $C(\emptyset) = \emptyset$, $A \subseteq C(A)$ for all $A \subseteq X$, $C(C(A)) = C(A)$, and $C(A \cup B) = C(A) \cup C(B)$ for all $A, B \subseteq X$.

Example 1.6. Problem 1.4 shows that if X is a topological space, then the closure operator defines a Kuratowski closure operator.

Problem 1.7. *Let X be a set with a Kuratorwski closure operator C. Say that $Z \subseteq X$ is C-closed if $C(Z) = Z$. Say that $U \subseteq X$ is C-open if $X \setminus U$ is closed. Let $\mathcal{U} \subseteq \mathbf{P}(X)$ be the subset of C-open subsets of X. Show that $\mathcal{U}$ defines a topology on X.*

Remark 1.8. Problem 1.7 shows that the notion of a topological space as we have studied it is equivalent to the notion of a set with a notion of ‘closure’ for its subsets. This is one of several equivalent notions of topological space which have slightly different flavors.

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